Math 2130
HW 2 Solutions

$\boxed{①(a)}$ $f(x,y) = xe^y$

$$\frac{\partial f}{\partial x} = 1 \cdot e^y = e^y$$

$$\frac{\partial f}{\partial y} = xe^y$$

$\boxed{①(b)}$ $f(x,y) = 4x^3y^2 + 3x^2y^3 + 10$

$$f_x = 12x^2y^2 + 6xy^3$$

$$f_y = 8x^3y + 9x^2y^2$$

$\boxed{①(c)}$ $g(x,y) = e^{x^2y}$

$$\frac{\partial g}{\partial x} = e^{x^2y} \cdot (2xy) = 2xye^{x^2y}$$

$$\frac{\partial g}{\partial y} = e^{x^2y} \cdot (x^2) = x^2e^{x^2y}$$

$\boxed{①(d)}$ $f(s,t) = \dfrac{s-t}{s+t}$

$\boxed{\text{Recall: } \left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}}$

$$f_s = \frac{(1-0)(s+t) - (s-t)(1+0)}{(s+t)^2} = \frac{2t}{(s+t)^2}$$

$$f_t = \frac{(0-1)(s+t) - (s-t)(0+1)}{(s+t)^2} = \frac{-2s}{(s+t)^2}$$

$\boxed{①(e)}$ $h(x,y) = x\cos(xy)$

$$\frac{\partial h}{\partial x} = 1\cdot\cos(xy) + x\cdot(-\sin(xy)\cdot y)$$

$$= \cos(xy) - xy\sin(xy)$$

$$\frac{\partial h}{\partial y} = x(-\sin(xy)\cdot x) = -x^2\sin(xy)$$

$\boxed{①(f)}$ $f(x,y) = x^3 y e^{2x^3}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^3y)\cdot e^{2x^3} + (x^3y)\cdot\frac{\partial}{\partial x}(e^{2x^3})$$

$$= 3x^2y\,e^{2x^3} + x^3y(e^{2x^3}\cdot 6x^2)$$

$$= 3x^2y\,e^{2x^3} + 6x^5y\,e^{2x^3}$$

$$\frac{\partial f}{\partial y} = x^3 e^{2x^3}$$

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2 (a) $f(x,y) = x^3 + xy^2 + 1$

$f_x = 3x^2 + y^2$

$f_y = 2xy$

$f_{xx} = 6x$

$f_{yy} = 2x$

$f_{xy} = 2y$

$f_{yx} = 2y$

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2 (b) $f(x,y) = x^3 e^{2y}$

$$\frac{\partial f}{\partial x} = 3x^2 e^{2y}$$

$$\frac{\partial^2 f}{\partial x^2} = 6xe^{2y}$$

$$\frac{\partial^2 f}{\partial y \partial x} = 3x^2(e^{2y} \cdot 2) = 6x^2 e^{2y}$$

$$\frac{\partial f}{\partial y} = x^3(e^{2y} \cdot 2) = 2x^3 e^{2y}$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^3(e^{2y} \cdot 2) = 4x^3 e^{2y}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 6x^2 e^{2y}$$

$\boxed{2)(c)}$ $f(x,y) = y\sin(xy)$

$f_x = y\cos(xy)\cdot y = y^2\cos(xy)$

$f_{xx} = y^2(-\sin(xy)\cdot y) = -y^3\sin(xy)$

$f_{xy} = 2y\cos(xy) + y^2(-\sin(xy)\cdot x)$

$= 2y\cos(xy) - y^2x\sin(xy)$

$f_y = 1\cdot\sin(xy) + y\cos(xy)\cdot x$

$= \sin(xy) + xy\cos(xy)$

$f_{yy} = \cos(xy)\cdot y + x\cos(xy) + xy(-\sin(xy)\cdot x)$

$= y\cos(xy) + x\cos(xy) - x^2y\sin(xy)$

$f_{yx} = \cos(xy)\cdot y + y\cos(xy) + xy(-\sin(xy)\cdot y)$

$= 2y\cos(xy) - xy^2\sin(xy)$

$\boxed{②\,(d)}$ $g(s,t) = \ln(st)$

$$\frac{\partial g}{\partial s} = \frac{1}{st} \cdot t = \frac{1}{s} = s^{-1}$$

$$\frac{\partial^2 g}{\partial s^2} = -s^{-2}$$

$$\frac{\partial^2 g}{\partial t \partial s} = 0$$

$$\frac{\partial g}{\partial t} = \frac{1}{st} \cdot s = \frac{1}{t} = t^{-1}$$

$$\frac{\partial^2 g}{\partial t^2} = -t^{-2}$$

$$\frac{\partial^2 g}{\partial s \partial t} = 0$$

3 $h(x,y,z) = \cos(x+y+z^2)$

$$\frac{\partial h}{\partial x} = -\sin(x+y+z^2)\cdot(1+0+0) = -\sin(x+y+z^2)$$

$$\frac{\partial^2 h}{\partial x \partial z} = -\cos(x+y+z^2)\cdot(0+0+2z)$$

$$= -2z\cos(x+y+z^2)$$

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$$h(x,y) = \sqrt{x^2+3y} = (x^2+3y)^{1/2}$$

$$h_x = \frac{1}{2}(x^2+3y)^{-1/2}(2x) = x(x^2+3y)^{-1/2}$$

$$h_{xx} = 1 \cdot (x^2+3y)^{-1/2} + x\left[-\frac{1}{2}(x^2+3y)^{-3/2} \cdot (2x)\right]$$

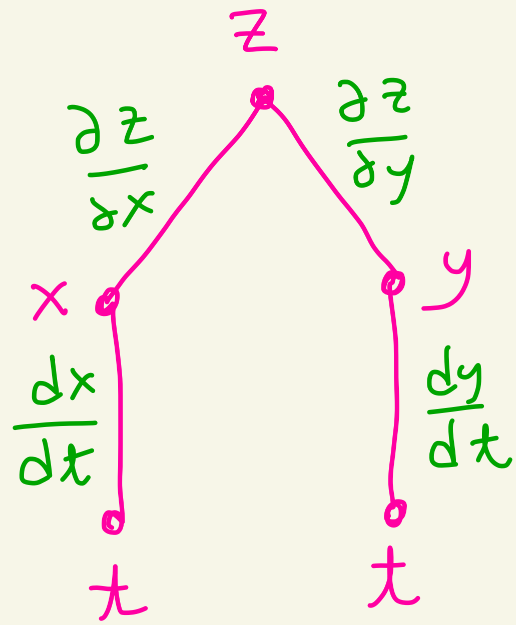
$$= (x^2+3y)^{-1/2} - x^2(x^2+3y)^{-3/2}$$

$$h_{xy} = x \cdot \left[-\frac{1}{2}(x^2+3y)^{-3/2} \cdot 3\right]$$

$$= -\frac{3}{2}x(x^2+3y)^{-3/2}$$

$\boxed{⑤}$

$$z = x\sin(y)$$
$$x = t^2$$
$$y = 4t^3$$



$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

$$= \left(\sin(y)\right)(2t) + \left(x\cos(y)\right)(12t^2)$$

$$= \left(\sin(4t^3)\right)(2t) + \left(t^2\cos(4t^3)\right)(12t^2)$$

$$= 2t\sin(4t^3) + 12t^4\cos(4t^3)$$

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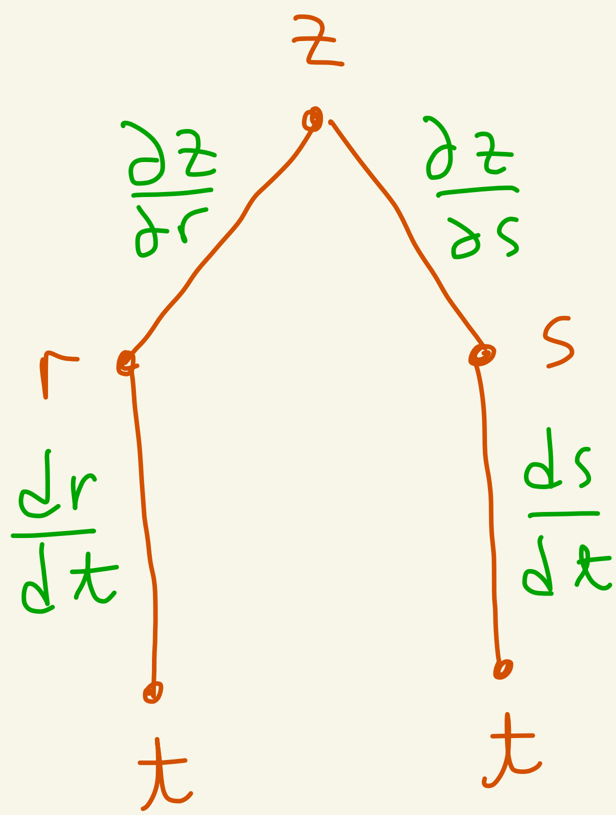
$$z = \sqrt{r^2 + s^2}$$

$$r = \cos(2t)$$

$$s = \sin(2t)$$



$$\frac{dz}{dt} = \frac{\partial z}{\partial r}\frac{dr}{dt} + \frac{\partial z}{\partial s}\frac{ds}{dt}$$

$$= \frac{1}{2}(r^2+s^2)^{-1/2}(2r)(-\sin(2t)\cdot 2)$$

$$+ \frac{1}{2}(r^2+s^2)^{-1/2}(2s)(\cos(2t)\cdot 2)$$

When $t = \pi$ we have

$$r = \cos(2\pi) = 1$$

$$s = \sin(2\pi) = 0$$

So at $t=\pi$ we have

$$\left.\frac{dz}{dt}\right|_{t=\pi} = \frac{1}{2}\left(1^2+0^2\right)^{-1/2}\cdot(2\cdot 1)\left(-\underbrace{\sin(2\pi)}_{0}\cdot 2\right)$$
$$+\frac{1}{2}\left(1^2+0^2\right)^{-1/2}\cdot\underbrace{(2\cdot 0)}_{0}\left(\cos(2\pi)\cdot 2\right)$$

$$= 0 + 0 = 0$$

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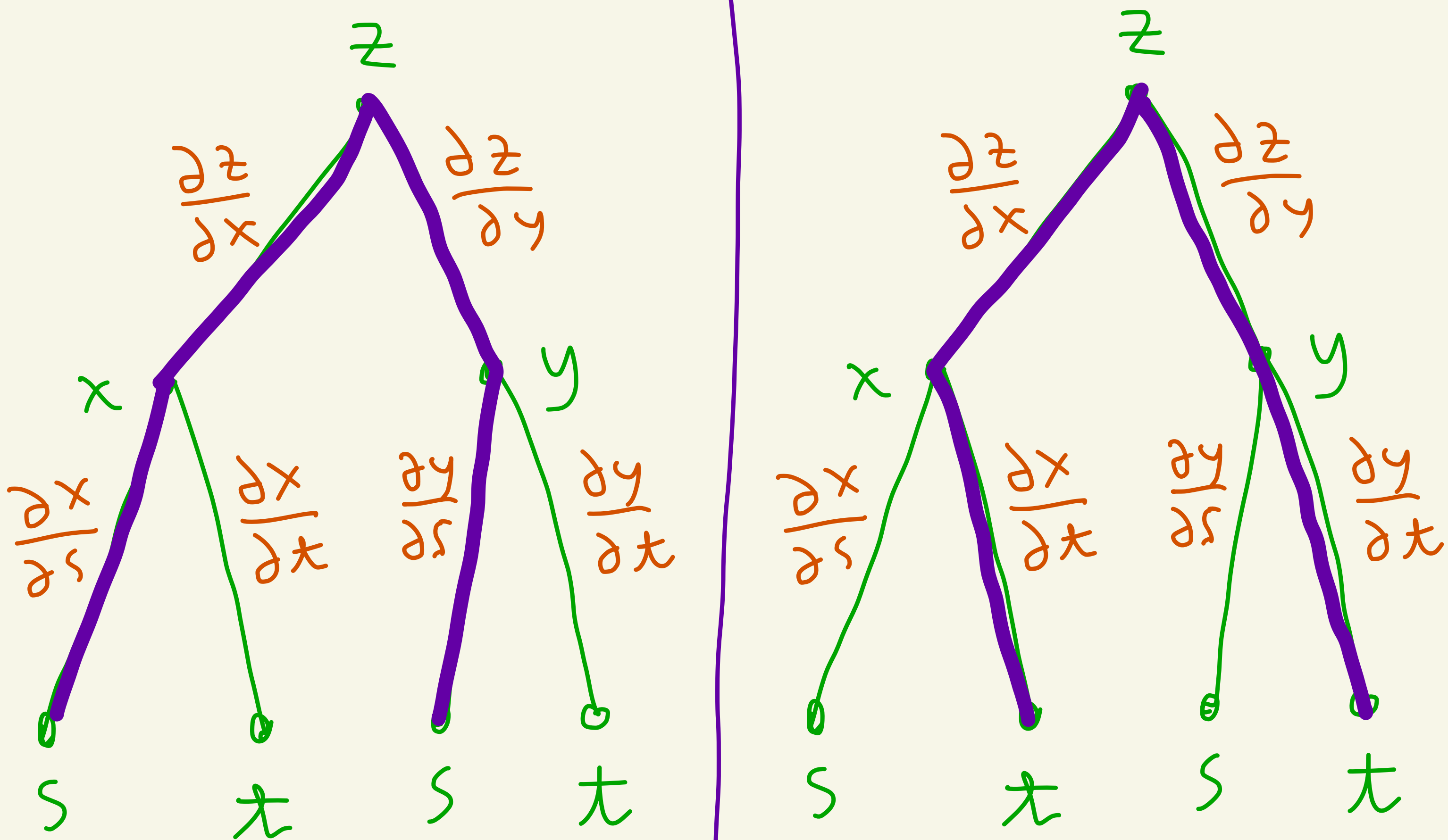
$z = \sin(2x + y)$

$x = s^2 - t^2$

$y = s^2 + t^2$



$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

We have

$$z_s = \frac{\partial z}{\partial s} = \overbrace{\Big(\cos(2x+y)\cdot 2\Big)}^{\partial z/\partial x}\overbrace{\Big(2s\Big)}^{\partial x/\partial s}$$

$$+ \underbrace{\Big(\cos(2x+y)\cdot 1\Big)}_{\partial z/\partial y}\underbrace{\Big(2s\Big)}_{\partial y/\partial s}$$

$$= 4s\cos(2x+y) + 2s\cos(2x+y)$$

$$= 4s\cos(2s^2-2t^2+s^2+t^2)$$

$$+ 2s\cos(2s^2-2t^2+s^2+t^2)$$

$$= 6s\cos(3s^2-t^2)$$

$$z_t = \frac{\partial z}{\partial t} = \overbrace{\Big(\cos(2x+y)\cdot 2\Big)}^{\partial z/\partial x}\overbrace{\Big(-2t\Big)}^{\partial x/\partial t}$$

$$+ \underbrace{\Big(\cos(2x+y)\cdot 1\Big)}_{\partial z/\partial y}\underbrace{\Big(2t\Big)}_{\partial y/\partial t}$$

$$= -4t\cos(2x+y) + 2t\cos(2x+y)$$

$$= -2t\cos(2s^2 - 2t^2 + s^2 + t^2)$$

$$= -2t\cos(3s^2 - t^2)$$

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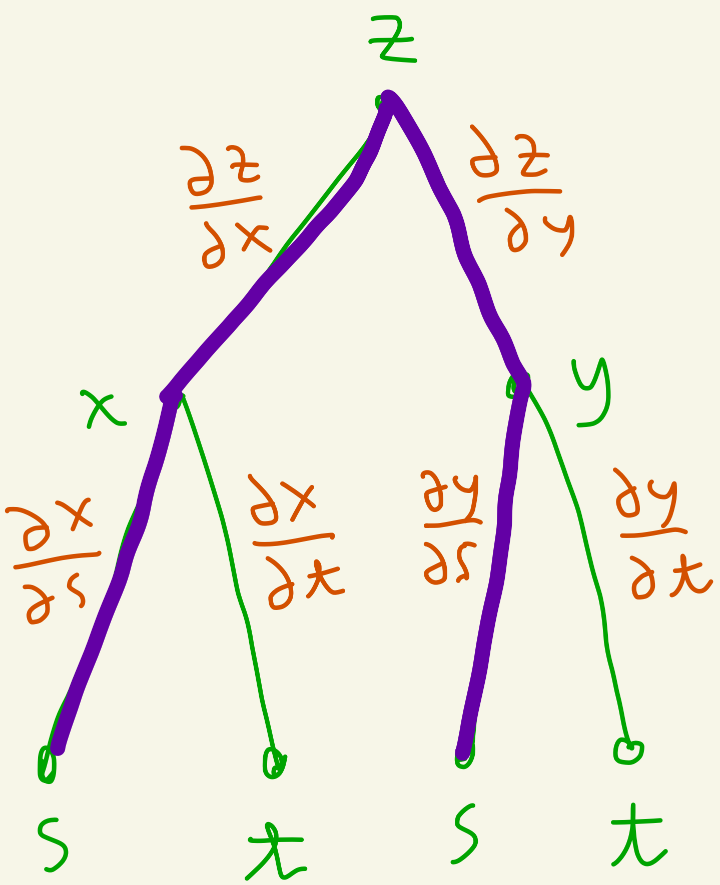
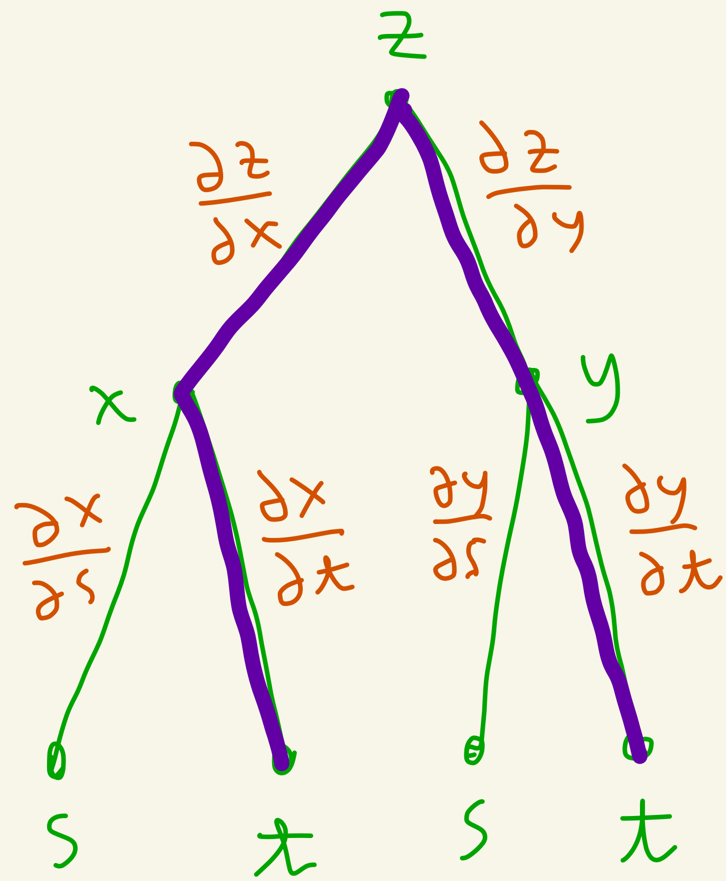
$$z = e^{x+y}$$

$$x = st$$

$$y = s+t$$





$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

We now evaluate $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ at $s=1$, $t=-1$.

Note when $s=1$ and $t=-1$ we have

$$x = st = (1)(-1) = -1$$

$$y = s+t = 1-1 = 0$$

Thus when $s=1$, $t=-1$ we have:

$$\frac{\partial z}{\partial s} = \underbrace{\left(e^{x+y}\right)}_{\partial z/dx}\underbrace{(t)}_{\partial x/\partial s} + \underbrace{\left(e^{x+y}\right)}_{\partial z/\partial y}\underbrace{(1)}_{\partial y/\partial s}$$

$$\left.\frac{\partial z}{\partial s}\right|_{\substack{s=1\\t=-1}} = \left(e^{-1+0}\right)(-1) + \left(e^{-1+0}\right)(1) = 0$$

$$\frac{\partial z}{\partial t} = \underbrace{\left(e^{x+y}\right)}_{\partial z/dx}\underbrace{(s)}_{\partial x/\partial t} + \underbrace{\left(e^{x+y}\right)}_{\partial z/\partial y}\underbrace{(1)}_{\partial y/\partial t}$$

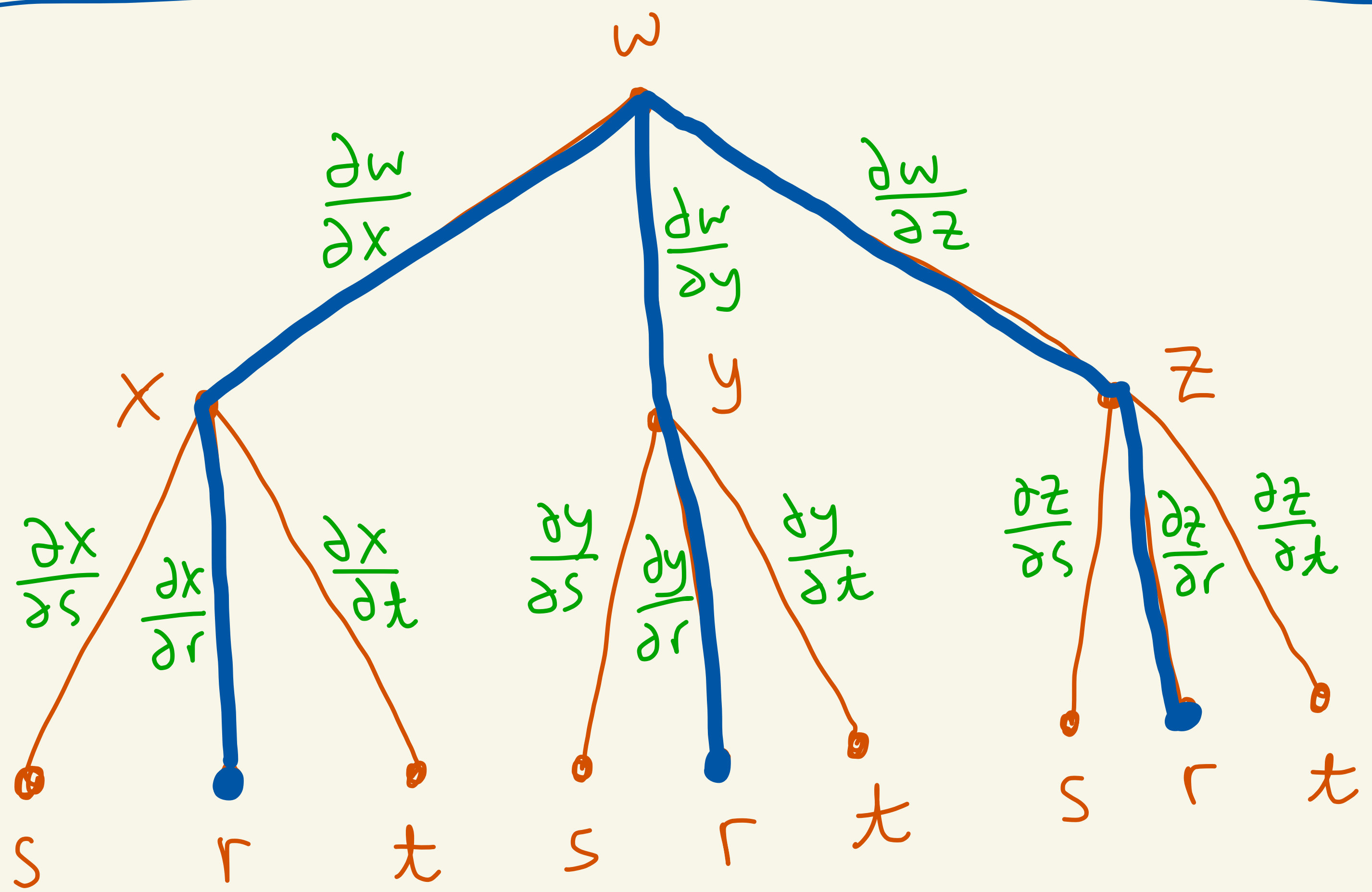
$$\left.\frac{\partial z}{\partial t}\right|_{\substack{s=1\\t=-1}} = \left(e^{-1+0}\right)(1) + \left(e^{-1+0}\right)(1) = 2e^{-1}$$

(9) $w = \sqrt{x^2 + 3y + z^3}$

$x = st$

$y = rs$

$z = rt$



This picture is for $\frac{\partial w}{\partial r}$ giving

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \frac{\partial z}{\partial r}$$

$w = (x^2 + 3y + z^3)^{1/2}, \quad x = st, \; y = rs, \; z = rt$

Thus,

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= \tfrac{1}{2}(x^2 + 3y + z^3)^{-1/2}(2x) \cdot (0)$$
$$+ \tfrac{1}{2}(x^2 + 3y + z^3)^{-1/2}(3) \cdot (s)$$
$$+ \tfrac{1}{2}(x^2 + 3y + z^3)^{-1/2}(3z^2)(t)$$

When $s = 1, r = 1, t = 0$ we get that $x = st = (1)(0) = 0$, $y = rs = (1)(1) = 1$,
$z = rt = (1)(0) = 0$

Thus,

$$\left.\frac{\partial w}{\partial r}\right|_{\substack{s=1\\r=1\\t=0}} = \tfrac{1}{2}(0^2 + 3(1) + 0^3)^{-1/2} \cdot (2 \cdot 0) \cdot (0)$$
$$+ \tfrac{1}{2}(0^2 + 3 \cdot 1 + 0^3)^{-1/2} \cdot (3)(1)$$
$$+ \tfrac{1}{2}(0^2 + 3 \cdot 1 + 0^3)^{-1/2} \cdot (3 \cdot 0^2)(0)$$
$$= \boxed{\frac{3}{2\sqrt{3}}} = \boxed{\frac{\sqrt{3}}{2}}$$